Lecture 15

Basics of Sets & Functions

What's a Set?

Definition: A **set** is an unordered collection of objects, called **elements** or members of the set. $a \in A$ denotes that a is a member of A, and $a \notin A$ denotes that a is not a member of A.

Examples:

$V = \{a, e, i, o, u\}$, is the set of vowels in English alphabet.

$E = \{2,4,6,8,10\}$, is the set of positive even integers ≤ 10.

Roster notation.

$E = \{x \mid x$ is a positive even integer $\leq 10\}$

$\mathbb{Q}^{+} = \{x \in \mathbb{R} \mid x = \frac{p}{q}$, for some positive integers p and $q\}$

Set builder notation.

Note: It is not necessary that members of a set should have a common property.
For instance, $\{99$, Bob, Jupiter$\}$ is a valid set.

More about Sets

Definition: Two sets are **equal** if and only if they have the same elements. In other words, if A and B are sets, then $A = B$ if and only if $\forall x(x \in A \iff x \in B)$.

Example: $\{1,2,3\} = \{1,3,2\}$ because they contain the same elements and the order does not matter. It also does not matter whether one element is listed more than once, therefore, $\{1,2,3,3,2,2\} = \{1,2,3\}$.

Definition: A set that contains no elements is called the **empty set** and denoted by $\varnothing$. A set with just one element is called a **singleton set**.

Note: Do not confuse $\varnothing$ with $\{\varnothing\}$, $\varnothing$ is the empty set and $\{\varnothing\}$ is a singleton set.

Russell's Paradox

Let's define a set S as

$$S = \{x \mid x \text{ is a set such that } x \notin x\}$$

It is reasonable to believe that an object either belongs to a set or not.

But,

$S \in S \;\rightarrow\; S \notin S$ (Assuming $S \in S$ lead to $S \notin S$, so $S \in S$ cannot be true.)

$S \notin S \;\rightarrow\; S \in S$ (Assuming $S \notin S$ lead to $S \in S$, so $S \notin S$ cannot be true.)

Problem lies in our intuitive notion of an object in the definition of Set.

- The theory that develops from this definition of set is called **Naive Set Theory**.
- **Axiomatic** set theories such as ZFC avoid these contradictions by having a set of axioms through which you can form a set.
- We will still continue with Naive Set Theory and avoid sets that can lead to contradictions.

Subsets and Cardinality

Definition: The set A is a **subset** of B iff every element of A is also an element of B.
$A \subseteq B$ denotes that A is a subset of B.

Proving $A \subseteq B, A \nsubseteq B, A = B$:

- To prove $A \subseteq B$, show that if $x \in A$, then $x \in B$.
- To prove $A \nsubseteq B$, find an x in A such that $x \notin B$.
- To prove $A = B$, show that $A \subseteq B$ and $B \subseteq A$.

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say that S is a finite set and that n is the **cardinality** of S. The cardinality of S is denoted by $|S|$. A set is said to be **infinite** if it is not finite.

Ordered Tuple

Definition: The **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, $\ldots$, and a_n as its nth element. Ordered 2-tuples are called **ordered pairs**.

Two ordered n-tuples, say $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, are equal if and only if each corresponding pair of their elements are equal, i.e., $a_i = b_i$ for $i = 1,2,\ldots,n$.

Cartesian Product

Definition: Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Example: Let $A = \{a, b\}$ and $B = \{1,2,3\}$. Then

$$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$$

$$B \times A = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$$

Definition: The cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i, for $i = 1,2,\ldots, n$.

Set Operations

Let A and B be two sets. Then the following operation can be defined on them,

Union: Denoted by $A \cup B$, is the set of all the elements that are either in A or B, or in both.

Intersection: Denoted by $A \cap B$, is the set of all the elements that are in both A and B.

A and B are **disjoint**, if $A \cap B = \varnothing$.

Difference: Denoted by $A - B$, is the set of all the elements that are in A but not in B.

Complement: Let U be the universal set. The complement of the set A, denoted by $\overline{A}$, is the complement of A with respect to U, i.e., $U - A$.

Note: Union and intersection of more than two sets defined as the natural extension of union and intersection of two sets.

Set Identities

Identity Laws:
$$A \cap U = A$$
$$A \cup \varnothing = A$$

Domination Laws:
$$A \cup U = U$$
$$A \cap \varnothing = \varnothing$$

Idempotent Laws:
$$A \cup A = A$$
$$A \cap A = A$$

Absorption Laws:
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

Complementation Law: $\overline{(\overline{A})} = A$

De Morgan's Laws:
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Complement Laws:
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

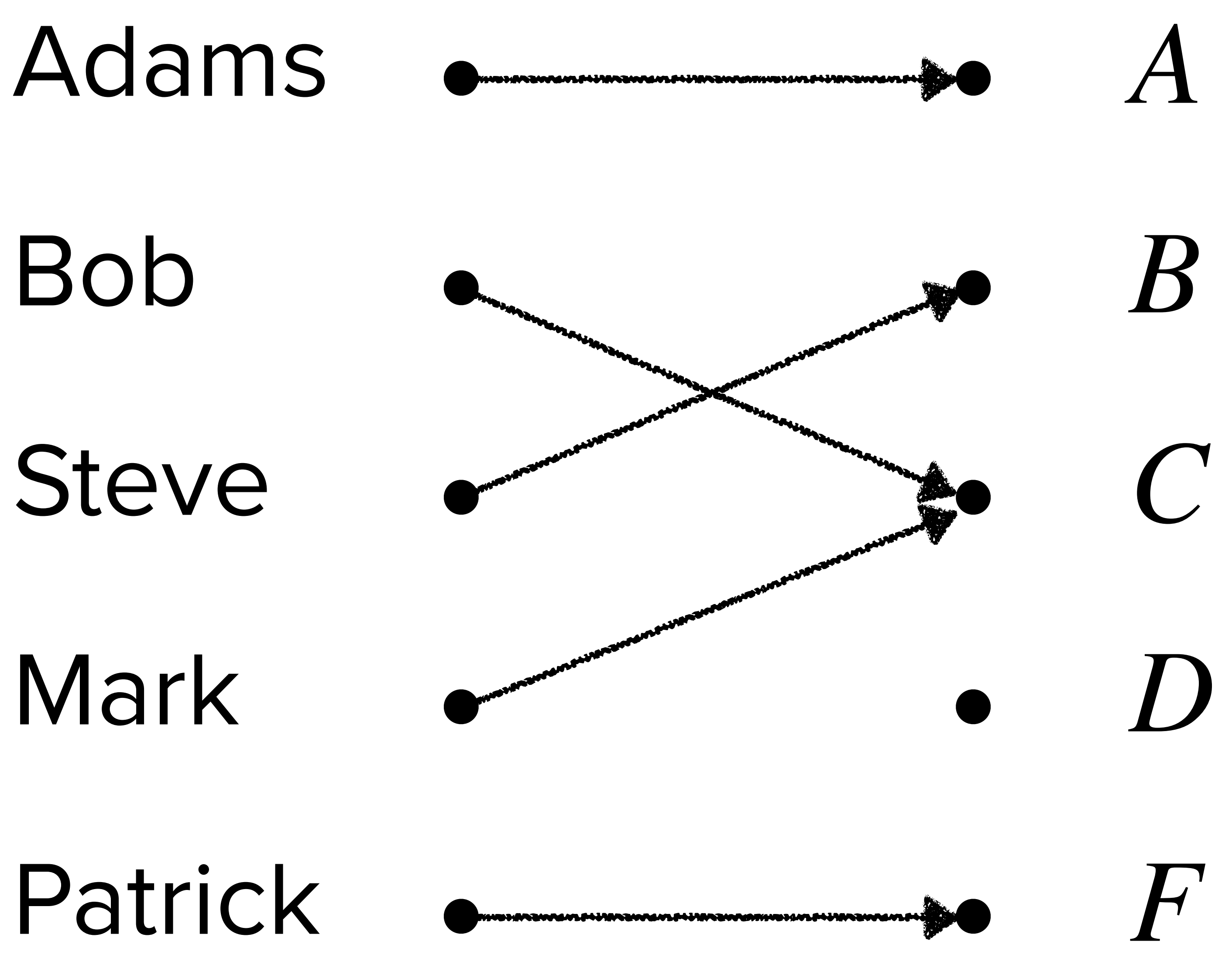
Commutative Laws:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws:
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

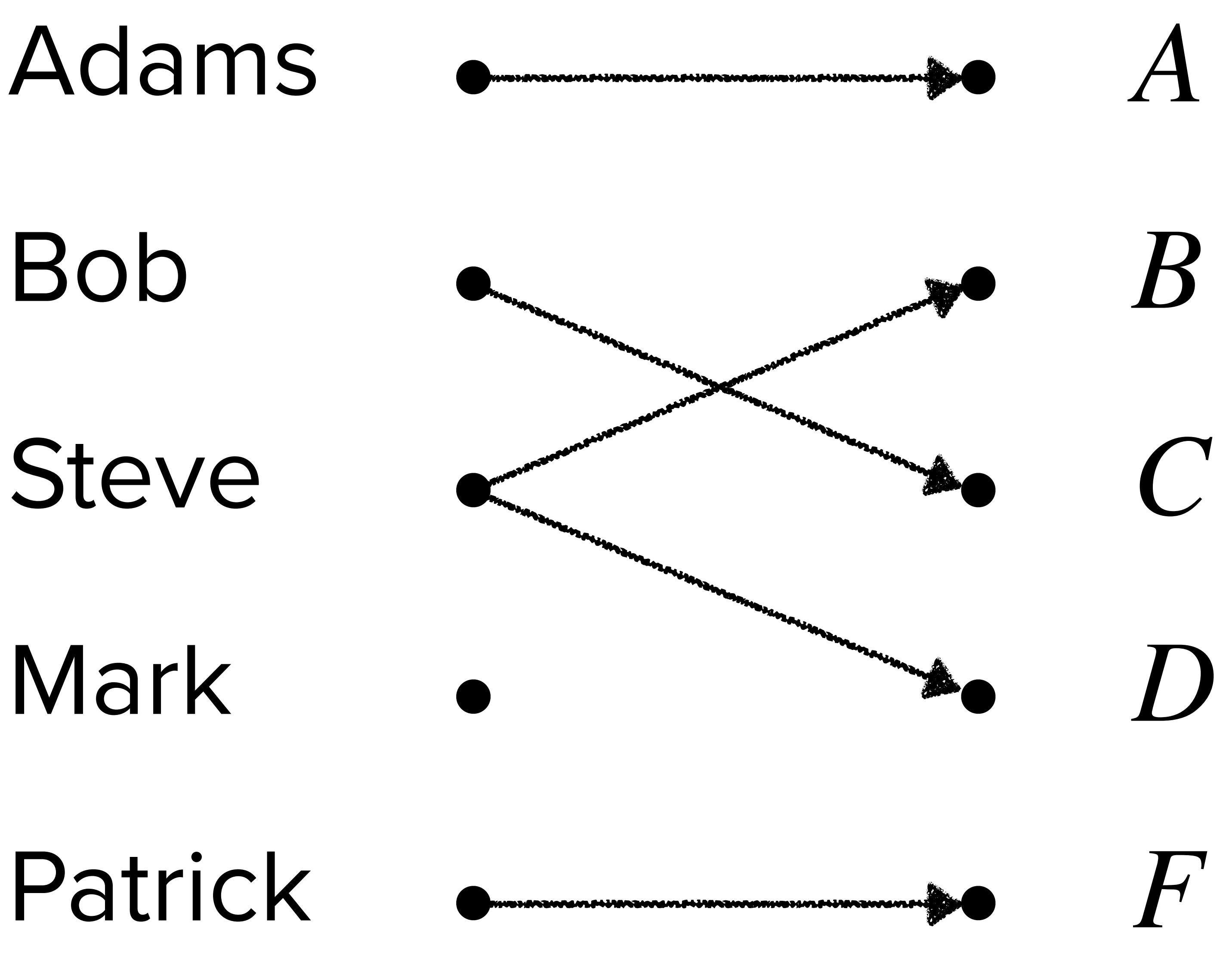
Distributive Laws:
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Functions

Definition: Let A and B be nonempty sets. A **function** f from A to B is an **assignment** of **exactly one element** of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$. A is called the **domain** of f and B is called the **image** or **range** of f.



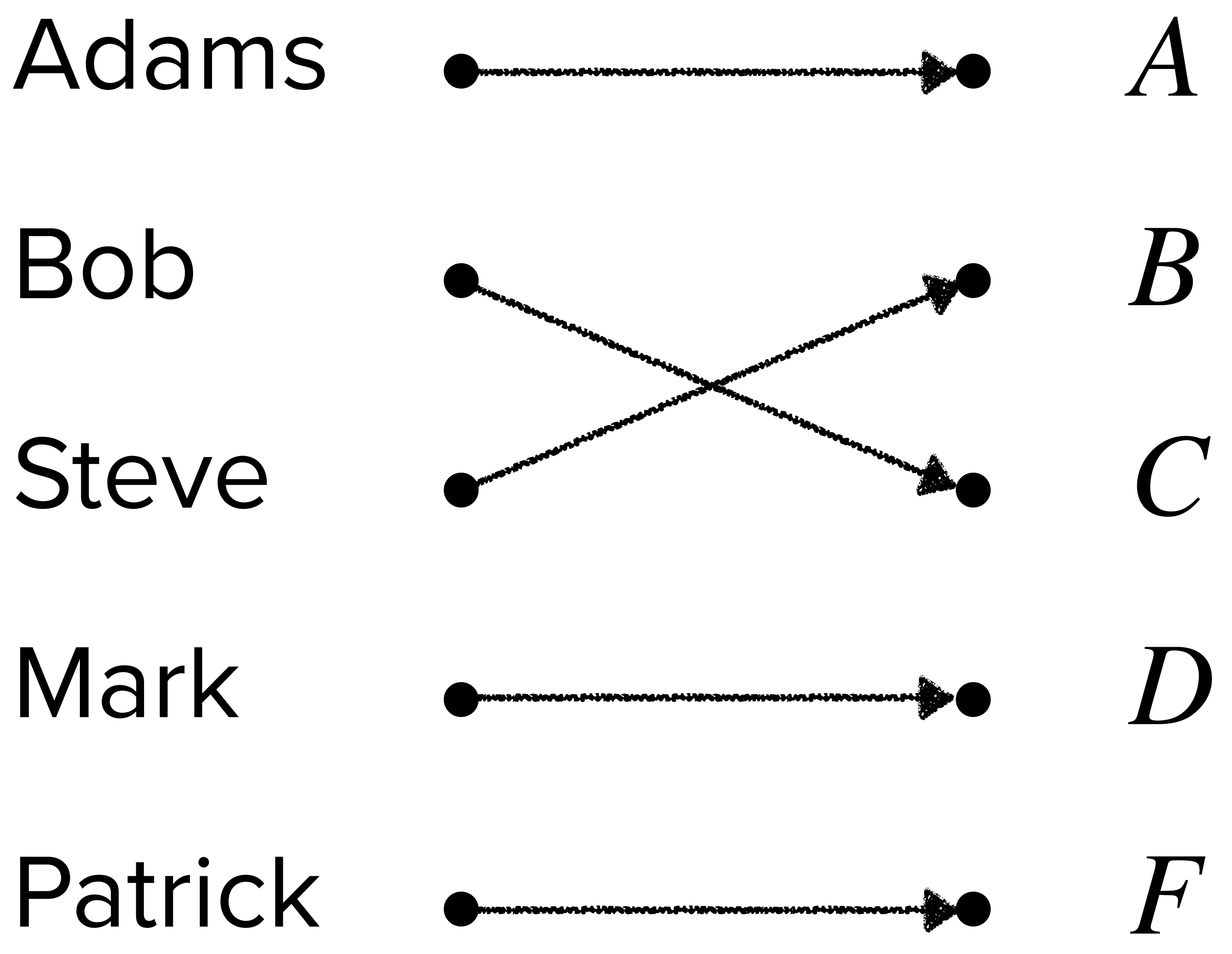
A function



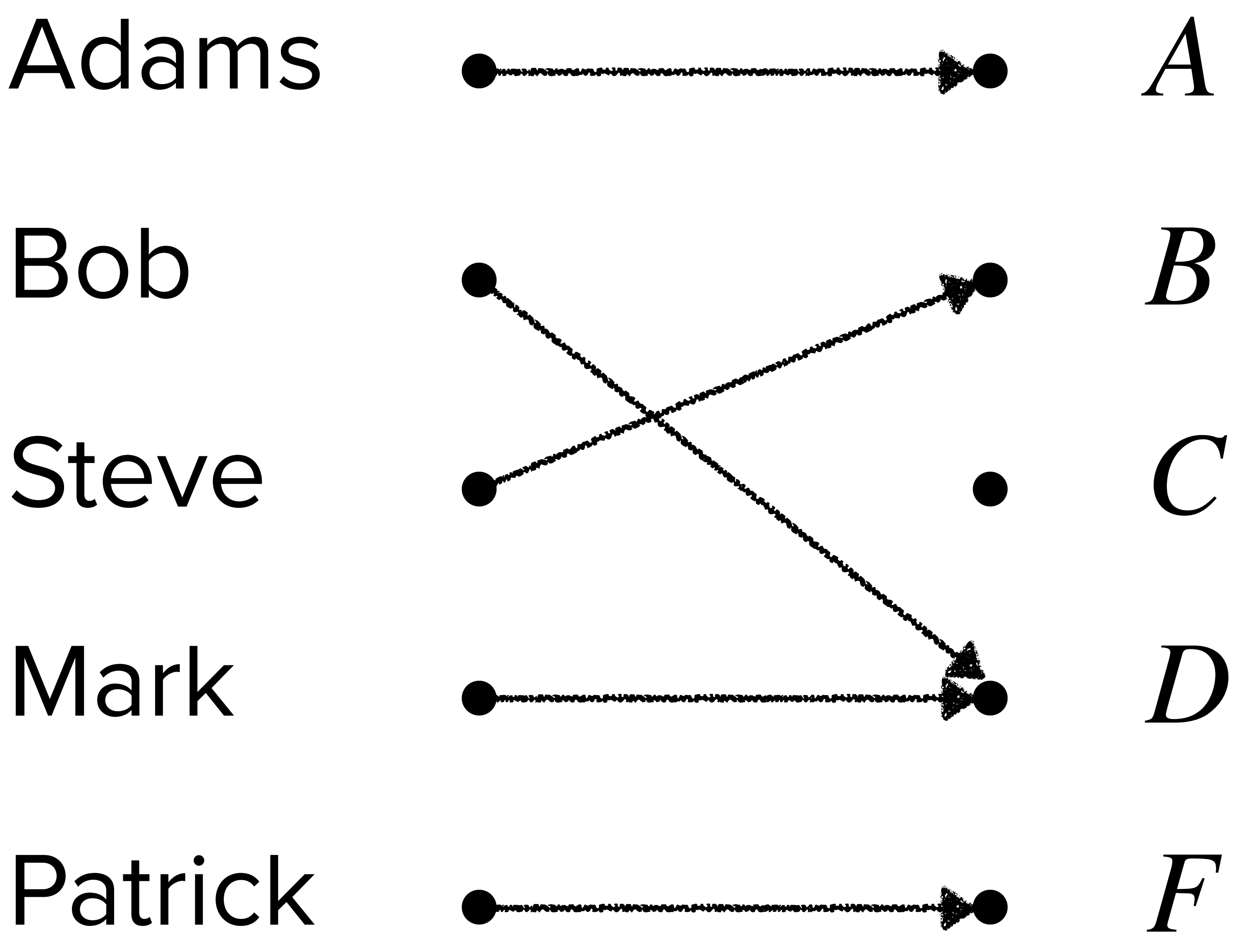
Not a function

One-to-One Functions

Definition: A function f is said to be **one-to-one** or an **injunction**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.



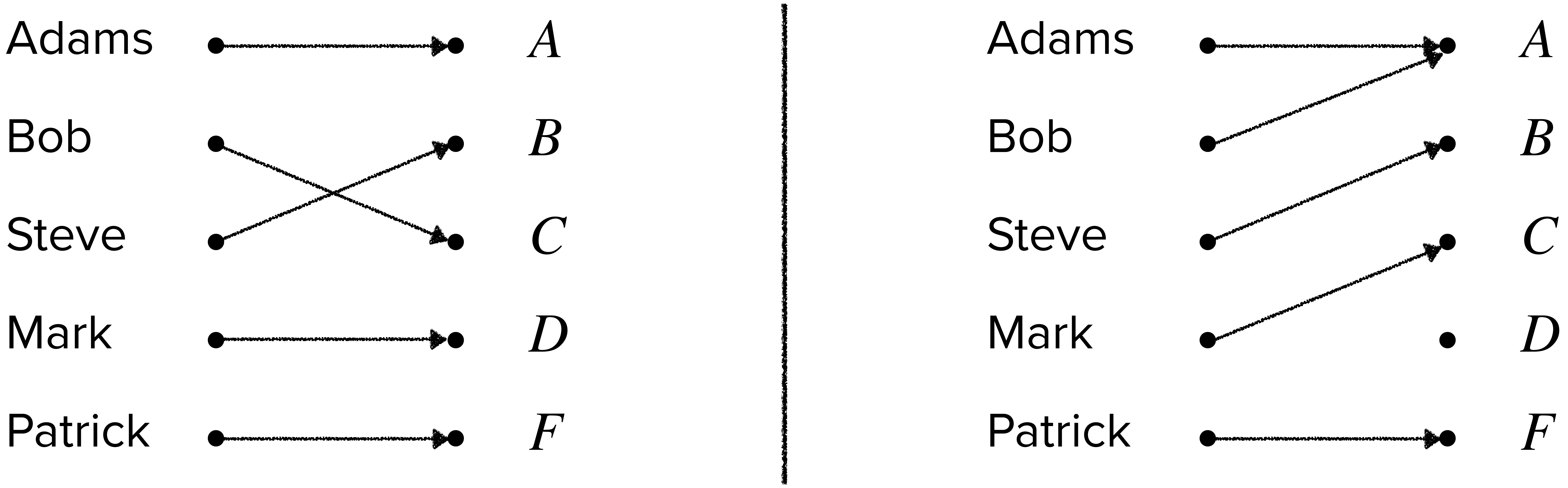
An injunction



Not an injunction

Onto Functions

Definition: A function f is said to be **onto** or a **surjection**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

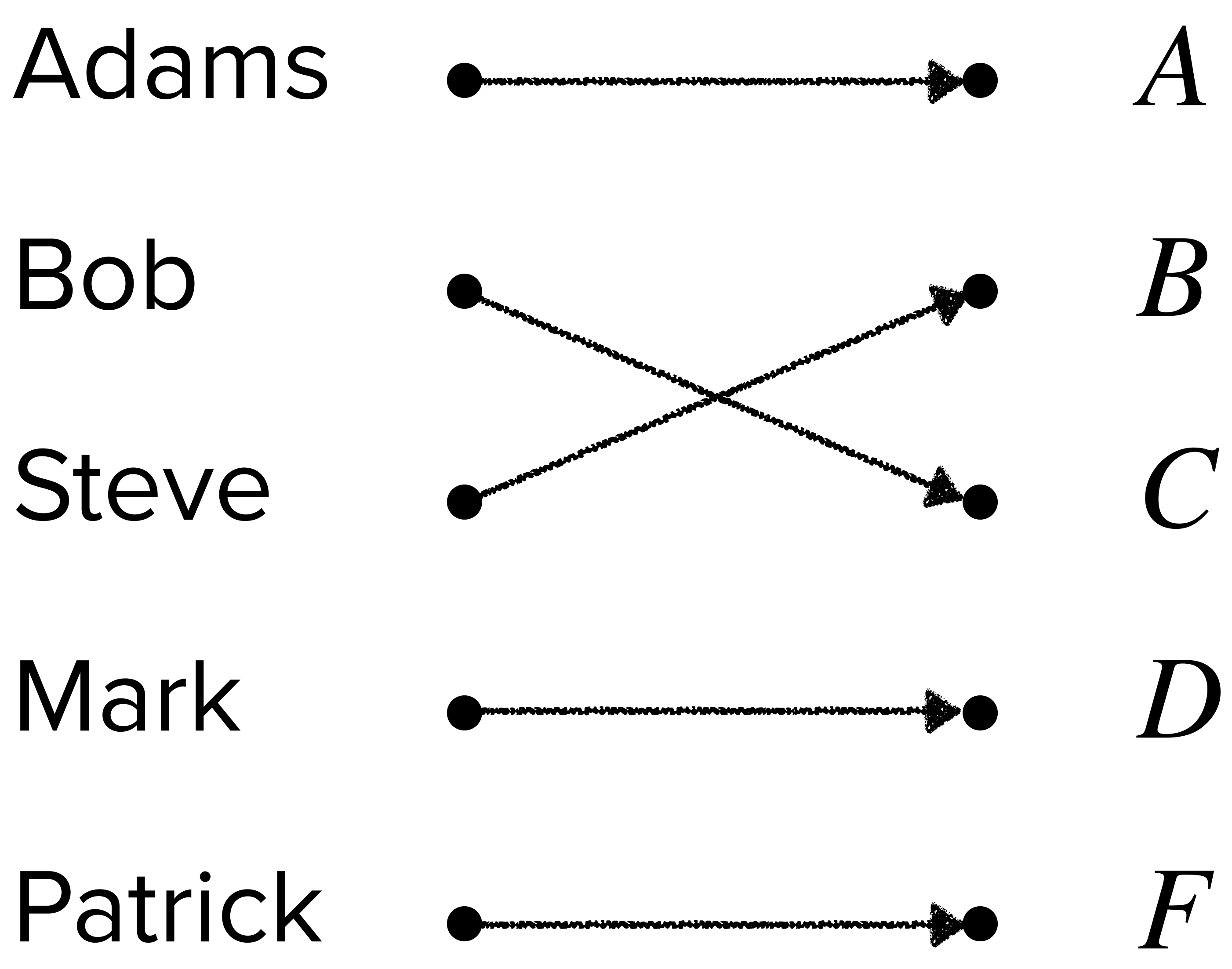


A surjection

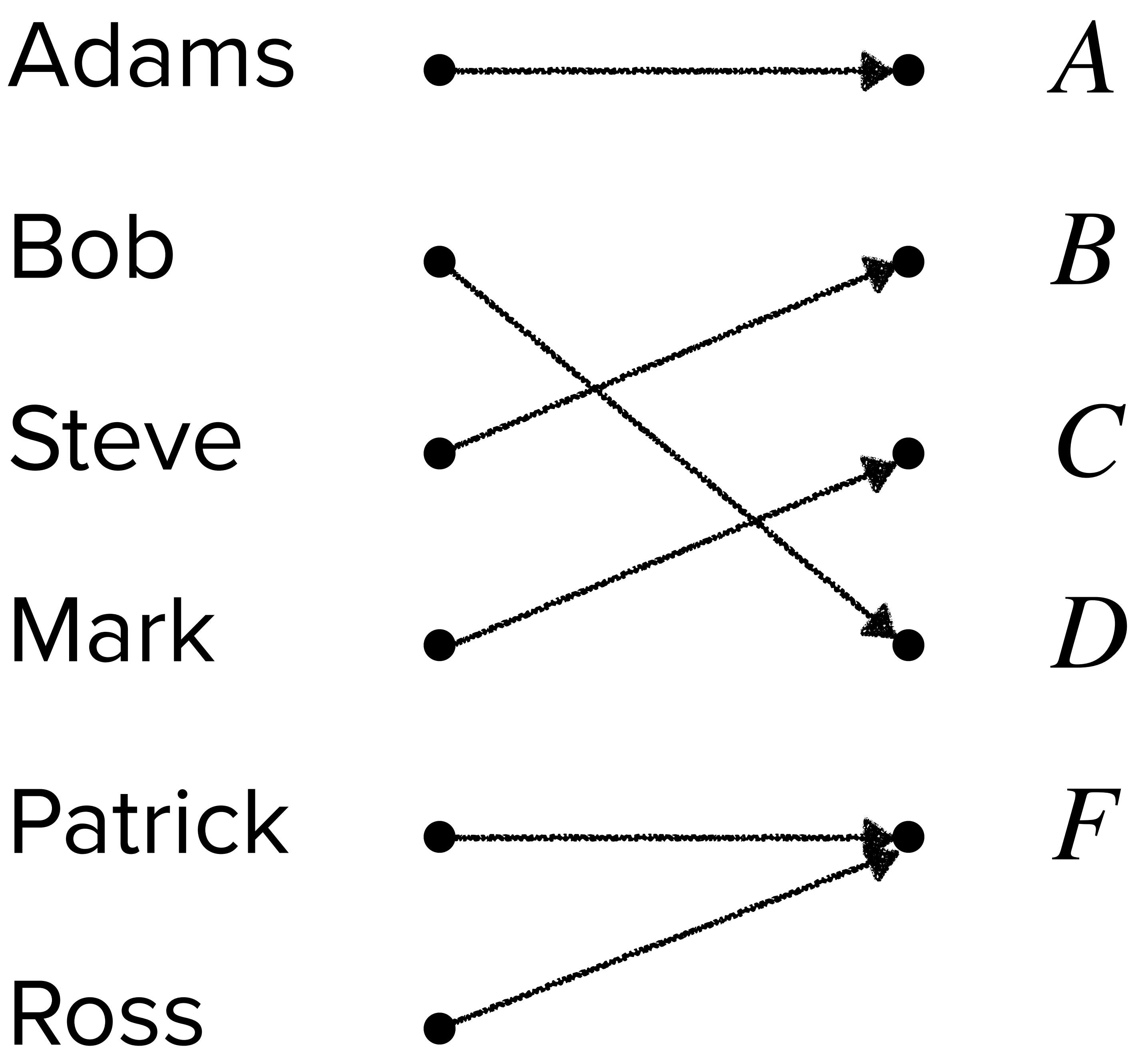
Not a surjection

Bijective Functions

Definition: A function f is said to be a **bijection**, if and only if it is both one-to-one and onto.



A bijection



Not a bijection

Inverse Function and Composition of Functions

Definition: Let f be a bijection from A to B. The **inverse function** of f, denoted by f^{-1}, is the function that assigns to an element b of B the unique element a in A such that $f(a) = b$, i.e, $f^{-1}(b) = a$.

Definition: Let g be a function from A to B and let f be a function from B to C. The **composition** of the functions f and g, denoted by $f \circ g$ for all $a \in A$, is defined by

$$(f \circ g)(a) = f(g(a))$$